

Solomon Friedberg - UMD Seminar

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## Whittaker ~~Eisen~~ Models for Metaplectic Groups and Crystal Graphs

(Joint with Dan Bump and Dan Bump)

(To appear in Annals of Math)

Langlands - Shahidi Theory say that the Whittaker

coefficients of Eisenstein series ~~can~~ can be expressed

as ~~the~~ products of L-functions, ~~for~~ for reductive groups.

Thus, we can use Whittaker coefficients to study L-functions.

Today: Study Whittaker coefficients of Eisenstein series on a cover of a reductive group. Describe

link between Whittaker coefficients of Eisenstein series and Representations of quantum groups, by means of twisted Euler products. (1)

Key idea: Whittaker coefficients are twisted Euler products (so outside Langlands conjectures) and their  $p$  parts are described by attaching number theoretic data to a given repn of a quantum group.

A twisted Euler product is, roughly, is a

Dirichlet series in complex variables  $s_1, \dots, s_r$ ,

$$D(s_1, \dots, s_r) = \sum \frac{H(c_1, \dots, c_r)}{|c_1|^{s_1} \cdots |c_r|^{s_r}}$$

where if  $(c_1 \cdots c_r, c'_1 \cdots c'_r) = 1$ , then

$$\cancel{H(c, c')} H(c, c'_1, \dots, c_r c'_r) = \mu_{c, c'} H(c_1, \dots, c_r) H(c'_1, \dots, c'_r)$$

where  $\mu_{c, c'}$  is an  $n^{\text{th}}$  root of unity.

and  $n$  is the degree of a cover  $\tilde{G}$  of  $G$ ,  $G$  a reductive group

Note: This allows one to reconstruct the series from

$$H(p^{k_1}, \dots, p^{k_r}), \quad p = \text{prime.}$$

(2)

## Basic Example : Kubota Dirichlet Series

Let  $n > 1$ ,  $n \in \mathbb{N}$ . Let  $F = \text{number field}$   
s.t.  $F$  contains the  $2n$ -th roots of unity

Let  $S :=$  sufficiently large finite set of places,  
containing all the infinite places.

Let  $F_S := \overline{\prod_{v \in S} F_v}$  ↪ completion of  $F$  at  $v$

Let  $\mathcal{O}_S :=$  ring of  $S$ -integers in  $F$ .

Sometimes we think of  $F$  as embedded diagonally  
in  $F_S$ ,  $F \hookrightarrow F_S$ .

Let  $\left(\frac{c}{d}\right)_n$  be an  $n^{\text{th}}$  power residue symbol.

Consider the  $n$ -fold cover

$$\tilde{SL}_2(F_S) \longrightarrow SL_2(F_S). \text{ This}$$

is given by some 2-cocycle

$$\sigma \in H^2(SL_2(F_S), \mu_n)$$

$\mu_n$  group of  $n^{\text{th}}$  roots of unity.

Lemma (Kubota):

Lemma (Kubota, and others): The map

$$X : \text{SL}_2(\mathcal{O}_S) \longrightarrow \mu_n$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{cases} \left( \frac{c}{d} \right)_n & \text{if } c \neq 0 \\ 1 & \text{if } c = 0 \end{cases}$$

This map splits the cocycle  
 $\sigma$  over  $\text{SL}_2(\mathcal{O}_S)$ .

Thus, we have an embedding, corresponding to this splitting,

$$\text{SL}_2(\mathcal{O}_S) \hookrightarrow \widetilde{\text{SL}}_2(F_S)$$

$$\gamma \longmapsto (\gamma, X(\gamma)).$$

(This is a homomorphism)

metaplectic

Consequence: We can form a  $\infty$  Eisenstein series  
~~(gen.)~~ (generalization of Eisenstein series from automorphic forms)

$$E(g, s, f_s) := \sum_{\gamma \in \mathcal{B}(O_s) \backslash SL_2(O_s)} f_s(i(\gamma)g)$$

$$\gamma \in \mathcal{B}(O_s) \backslash SL_2(O_s)$$

$\mathcal{B}$ orel in  $SL_2(O_s)$

where  $f_s \in \pi(s) := \{ f_s : SL_2(F_s) \xrightarrow{\sim} \mathbb{C} \}$

$$f\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} g\right) = |t_1|^{2s} f_s(g)$$

$$\forall t_i \in O_s^\times (F_s^\times)^n \}$$

Thm (Kubota):  $E(g, s, f_s)$  has analytic continuation

and functional equation under  $s \mapsto 1-s$ .

Consequence: The Fourier coefficients of  $E(g, s, f_s)$   
also have analytic continuation.

What are these Fourier coefficients?

Answer : They are infinite sums of Gauss sums.

Let  $\psi$  be an additive character of conductor  $\mathcal{O}_S$ . We have  $\mathfrak{g}$  Gauss sums

$$g(m, c) = \sum_{\substack{a \text{ s.t.} \\ (a, c) = 1}} \left(\frac{a}{c}\right)_n \psi\left(\frac{ma}{c}\right)$$

$\downarrow$   $m$  is in  $\mathcal{O}_S - \mathfrak{d}_{\mathcal{O}_S}$

Then Kubota shows that the  $m^{\text{th}}$  ~~of~~ Fourier coefficient ~~of~~ are

$$\sum_{\substack{\text{of } c, \\ c \text{ an ideal}}} \frac{g(m, c) \psi_{f_{S, m}}(c)}{|N(c)|^{2s}}$$

$\uparrow$   
norm

where  $\psi_{f_{S, m}}$  runs over a finite dim'l

vector space of functions on  $\mathbb{F}_S$ .  
 $g$  is the arithmetic part.

Then by the Chinese Remainder Theorem,

if  $(c, c') = 1$ ,  
$$g(m, cc') = \left(\frac{c}{c'}\right)_n \left(\frac{c'}{c}\right)_n g(m, c) g(m, c')$$

Also, if  $(m, c) = 1$ , then

$$g(m m, c) = \left(\frac{m}{c}\right)_n^{-1} g(m, c)$$

So this is "twisted multiplication in  $m$ ".

Question: What is the  $p$  part of the Kubota Dirichlet series?

Answer:  $g(p^a, p^b)$ , where  $a$  is fixed and  $b$  is varying.

~~But~~ But  $g(p^a, p^b) = 0$  if  $b \geq a+2$  and  ~~$g(p^a, p^{a+1})$~~

and  $g(p^a, p^{a+1})$  is always non-zero

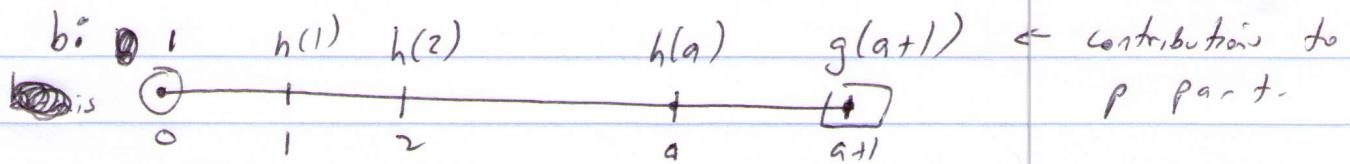
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$g(a+1)$

Also, if  $b \leq a$ ,  $g(p^a, p^b) = \begin{cases} \phi(p^b) & \text{if } n \mid b \\ 0 & \text{if } n \nmid b \end{cases}$

If  $b \leq a$ , we write  $\otimes h(b) := g(p^a, p^b)$

Picture: Recall,  $a$  is fixed



Key point: This is a crystal graph associated to a repn of quantum of  $SL_2$ , where each vertex represents a basis element, and the ~~bad edges~~ lines connecting the vertices correspond to Kashiwara operators.

The box  $\square$  corresponds to maximal path component to lowest weight vector.

The circle  $\circ$  corresponds to maximal path component to highest weight vector.

Generalization:

Thm (BBF): The  $m = (m_1, \dots, m_r)$ -th Fourier coefficient of the Borel Eulerian series on the  $n$ -fold cover of  $SL_{r+1}(F_\infty)$  is ~~(is special)~~

a special function times a series, where the

series is

$$\sum \frac{H_m(c_1, \dots, c_r) \psi(c_1, c_r)}{|c_1|^{2r} \dots |c_r|^{2r}}$$

where  $\psi$  runs over a finite dimensional vector space of functions on



$$(F_s^x)^r$$

and

- $H_m$  is twisted multiplicative in the  $c_i$  and the  $m_i$ .

- The  $p$ -coefficient is

$$H_{(p^{k_1}, \dots, p^{k_r})}(p^{k_1}, \dots, p^{k_r})$$
 is described

as follows : Let  $\epsilon_1, \dots, \epsilon_r$  be the fundamental weights, and

$\lambda = l_1 \epsilon_1 + \dots + l_r \epsilon_r$ . Look at the repn of the quantized repn of

the universal enveloping algebra,

$U_q(\mathfrak{gl}_{r+1}(C))$  of highest weight

$\lambda + \rho$ ; it has

a crystal graph  $\mathcal{B}_{\lambda+\rho} \leftarrow$  language from Quantum groups

Factor the long element  $\sigma$  of the Weyl group

$w_0 = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_N}$  into simple reflections (certain factorization is needed)

where  $w_{\alpha_i}$  are simple reflections; use to

assign to each vertex  $v \in \mathcal{B}_{\lambda+\rho}$   $a_1$

$N$ -tuple of non-negative integers

$$\mathcal{B} \models L(v) = (a_1, \dots, a_N)$$



Bernstein - Zelevinsky - Littleman path.

$$\text{Define } G(v) := \prod_{i=1}^N \left\{ \begin{array}{ll} g(a_i) & \text{if } a_i \text{ is maximal} \\ \text{Norm}(p)^{a_i} & \text{if } a_i \text{ is } \cancel{\text{maximal}} \text{ for this segment to lowest weight vector} \\ h(a_i) & \text{if } a_i \text{ is } \cancel{\text{maximal}} \text{ for highest weight vector neither} \\ 0 & \text{if both} \end{array} \right.$$

Then ~~we~~ ~~will~~

$$H_{(\rho^{k_1}, \dots, \rho^{k_r})} =$$

$$\sum_{v \in B_{\lambda+\beta}} G(v) \quad \text{~~weight~~}$$

$$\sum_{i=1}^r k_i x_i = \lambda + \beta - w_0(\text{weight } v)$$

This finishes the statement of the Theorem.